Homework 7 – Solutions

① $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ $\quad (A - \lambda I) = \begin{bmatrix} 2-\lambda & 3 \\ 0 & -1-\lambda \end{bmatrix}$

want $A = PDP^{-1}$

1) find eigenvalues

$$0 = \det(A - \lambda I) = (2-\lambda)(-1-\lambda) - 0(3)$$
$$= (2-\lambda)(-1-\lambda)$$

$\lambda = -1$ and 2

2) find eigenvectors

if $\lambda = -1$, $A - \lambda I = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

Row reduce $\begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ solved $b = -a$ $\quad$ basis $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

if $\lambda = 2$, $A - \lambda I = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}$

row reduce $\begin{bmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $\quad$ b=0 $\quad$ basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3) Construct P $\quad P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

4) Construct D $\quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

thus $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

$A^k = PDP^{-1} \cdot PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^k & (-1)^{k+1} + 2^k \\ 0 & (-1)^{k+1} \end{bmatrix}$

(2) $y = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$ $u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

a) $u_1 \cdot u_2 = 3(-4) + 4(3) + 0(0) = -12 + 12 = 0$

orthogonal ✓

b) $\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$

$= \frac{30}{25} u_1 + \frac{-15}{25} u_2$

$= \frac{6}{5} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$